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On the construction of L -equienergetic graphs

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Abstract

For a graph G with n vertices and m edges, and having Laplacian spectrum $\mu_1, \mu_2, \dots, \mu_n$ and signless Laplacian spectrum $\mu_1^+, \mu_2^+, \dots, \mu_n^+$, the Laplacian energy and signless Laplacian energy of G are respectively, defined as $LE(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$ and $LE^+(G) = \sum_{i=1}^n |\mu_i^+ - \frac{2m}{n}|$. Two graphs G_1 and G_2 of same order are said to be L -equienergetic if $LE(G_1) = LE(G_2)$ and Q -equienergetic if $LE^+(G_1) = LE^+(G_2)$. The problem of constructing graphs having same Laplacian energy was considered by Stevanovic for threshold graphs and by Liu and Liu for those graphs whose order is $n \equiv 0 \pmod{7}$. We consider the problem of constructing L -equienergetic graphs from any pair of given graphs and we construct sequences of non-cospectral (Laplacian, signless Laplacian) L -equienergetic and Q -equienergetic graphs from any pair of graphs having same number of vertices and edges. © 2015 Kalasalingam University. Production and Hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

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1. Introduction

Let G be a finite, simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. Throughout this paper, we denote such a graph by $G(n, m)$. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$ -square matrix of order n whose (i, j) -entry is equal to one if v_i is adjacent to v_j and equal to zero, otherwise. The adjacency spectrum of G is the spectrum of its adjacency matrix. The energy of G is the sum of the absolute values of the adjacency eigenvalues of G (see [1]). This quantity introduced by Gutman has noteworthy chemical applications (see [2]).

Let $D(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix associated to G , where d_i is the degree of vertex v_i . The matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are called Laplacian and signless Laplacian matrices and the spectrum of the matrices $L(G)$ and $L^+(G)$ are called Laplacian spectrum (L -spectrum) and signless Laplacian spectrum (Q -spectrum) of G , respectively. Being real symmetric, positive semi-definite matrices, we take $0 = \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_1$ and $0 \leq \mu_n^+ \leq \mu_{n-1}^+ \leq \dots \leq \mu_1^+$ to be the L -spectrum and Q -spectrum of G , respectively. It is well known that $\mu_n = 0$ with multiplicity equal to the number of connected components of G (see [3]). Fiedler [3] showed that a graph G is connected if and only if its second smallest Laplacian eigenvalue is

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positive and called this as the algebraic connectivity of the graph G . Also it is well known that for a bipartite graph the L -spectra and Q -spectra coincide (see [4]). The Laplacian energy of a graph G as put forward by Gutman and Zhou (see [5]) is defined as $LE(G) = \sum_{i=1}^n |\mu_i - \frac{2m}{n}|$. This quantity, which is an extension of graph-energy concept has found remarkable chemical applications beyond the molecular orbital theory of conjugated molecules (see [6]). In analogy to Laplacian energy, the signless Laplacian energy of G is defined as $LE^+(G) = \sum_{i=1}^n |\mu_i^+ - \frac{2m}{n}|$. Both these graph-energy extensions have been extensively studied and can be seen in the literature (see [7–11] and the references therein). More results can be seen in [12–17]. It is easy to see that $tr(L(G)) = \sum_{i=1}^n \mu_i = \sum_{i=1}^{n-1} \mu_i = 2m$ and $tr(LE^+(G)) = \sum_{i=1}^n \mu_i^+ = 2m$.

Two graphs G_1 and G_2 of the same order are said to be equienergetic if $E(G_1) = E(G_2)$, (see [18,19]). In analogy to this, two graphs G_1 and G_2 of the same order are said to L -equienergetic if $LE(G_1) = LE(G_2)$ and Q -equienergetic if $LE^+(G_1) = LE^+(G_2)$. In [20,21], the families of L -equienergetic graphs have been constructed for a particular class of graphs. As per the existing literature the problem of constructing L -equienergetic graphs from a general pair of given graphs is still open and the present work is aimed in this direction. In this paper, we will show how sequences of L -equienergetic (Q -equienergetic) non-cospectral (Laplacian, signless Laplacian) graphs can be constructed from any pair of graphs having the same number of vertices and edges.

We denote the complement of a graph G by \bar{G} , the complete graph on n vertices by K_n , the empty graph by \bar{K}_n and the complete bipartite graph with cardinalities of partite sets as q and r by $K_{q,r}$. The rest of the paper is organized as follows. In Section 2, some preliminary results which are important throughout the paper are presented. In Section 3, construction of families of L -equienergetic (Q -equienergetic) graphs by using the graph operations like, union, join and complement is given and finally in Section 3, product of graphs is used for the construction of L -equienergetic (Q -equienergetic) graphs.

2. Preliminaries

In this section, we define some graph operations with terminology from (see [22]) together with their L -spectra and Q -spectra (see [23]) which are used throughout the paper.

For the graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$, the Cartesian product is a graph $G = G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge $((u_1, v_1), (u_2, v_2))$ if and only if $u_1 = u_2$ and (v_1, v_2) is an edge of G_2 , or $v_1 = v_2$ and (u_1, u_2) is an edge of G_1 . The following observation [23] gives the L -spectra (Q -spectra) of the Cartesian product of graphs.

Lemma 2.1. *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having L -spectra (Q -spectra) $\mu_1, \mu_2, \dots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2}$, respectively, then the L -spectra (Q -spectra) of $G = G_1 \times G_2$ is $\mu_i + \sigma_j$ where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

The conjunction (Kronecker product) of G_1 and G_2 is a graph $G = G_1 \otimes G_2$ with vertex set $V(G_1) \times V(G_2)$ and an edge $((u_1, v_1), (u_2, v_2))$ if and only if (u_1, u_2) and (v_1, v_2) are edges in G_1 and G_2 , respectively. The following result [23] gives the L -spectra (Q -spectra) of the Kronecker product of graphs.

Lemma 2.2. *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having L -spectra (Q -spectra) $\mu_1, \mu_2, \dots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2}$, respectively, then the L -spectra (Q -spectra) of $G = G_1 \otimes G_2$ is $\mu_i \sigma_j$, where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

The join (complete product) of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2 . The L -spectra of join of graphs is given by the following result [23].

Lemma 2.3. *If $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ are two graphs having L -spectra respectively as $\mu_1, \mu_2, \dots, \mu_{n_1-1}$, $\mu_{n_1} = 0$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2-1}, \sigma_{n_2} = 0$, then the L -spectra of $G = G_1 \vee G_2$ is $n_1 + n_2, n_1 + \sigma_1, n_1 + \sigma_2, \dots, n_1 + \sigma_{n_2-1}, n_2 + \mu_1, n_2 + \mu_2, \dots, n_2 + \mu_{n_1-1}, 0$.*

The disjoint union of graphs G_1 and G_2 is a graph G with vertex set and edge set, respectively, equal to union of vertex sets and edge sets of G_1 and G_2 . The complement of a graph G is the graph \bar{G} with vertex set same as G and with two vertices adjacent in \bar{G} if and only if they are non-adjacent in G . Lemmas 2.4 and 2.5 (see [23]) give the spectra of disjoint union of graphs and L -spectra of the complement of a graph.

Lemma 2.4. If G_1 and G_2 are two vertex disjoint graphs, then the spectra (adjacency, Laplacian, signless Laplacian) of graph $G = G_1 \cup G_2$ is the union of spectra of G_1 and G_2 .

Lemma 2.5. If $G(n, m)$ is a graph having L -spectra $\mu_1, \mu_2, \dots, \mu_n = 0$, then the L -spectra of complement \bar{G} of G is $n - \mu_1, n - \mu_2, \dots, n - \mu_{n-1}, \mu_n = 0$.

If G is any proper subgraph of K_n , then the L -spectra of G (see [24]) is given by the following result.

Lemma 2.6. Let $G(s, m)$ be a subgraph of K_n and having L -spectra μ_1, \dots, μ_s , then the L -spectra of the graph $K_n - E(G)$ is $n - \mu_1, n - \mu_2, \dots, n - \mu_s, n((n - s - 1)\text{-times}), 0$.

3. Construction of L -equienergetic (Q -equienergetic) graphs by means of operations, union, join and complement

In this section, we will construct sequences of L -equienergetic and Q -equienergetic non-cospectral (Laplacian, signless Laplacian) graphs from a given pair of connected graphs having same number of vertices and edges by using the well known graph operations like disjoint union, join and complement of graphs.

From a given pair of connected graphs, the following result constructs a sequence of L -equienergetic graph pairs which are disconnected.

Theorem 3.1. Let G_1 and G_2 be two connected graphs having L -spectra $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$, respectively. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(G_1 \cup \bar{K}_p) = LE(G_2 \cup \bar{K}_p)$.

Proof. By Lemma 2.4, the L -spectra of the graphs $G_1 \cup \bar{K}_p$ and $G_2 \cup \bar{K}_p$ are respectively $\mu_1, \mu_2, \dots, \mu_{n-1}, 0$ ($(p+1)$ -times) and $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, 0$ ($(p+1)$ -times), with average vertex degree $\frac{2m'}{n'} = \frac{2m}{p+n}$. Therefore,

$$\begin{aligned} LE(G_1 \cup \bar{K}_p) &= \sum_{i=1}^{n-1} \left| \mu_i - \frac{2m'}{n'} \right| + (p+1) \left| 0 - \frac{2m'}{n'} \right| \\ &= \sum_{i=1}^{n-1} \left(\mu_i - \frac{2m}{n+p} \right) + (p+1) \frac{2m}{p+n} \\ &= 2m + (p-n-2) \frac{2m}{p+n}. \end{aligned}$$

Also,

$$\begin{aligned} LE(G_2 \cup \bar{K}_p) &= \sum_{i=1}^{n-1} \left| \lambda_i - \frac{2m'}{n'} \right| + (p+1) \left| 0 - \frac{2m'}{n'} \right| \\ &= \sum_{i=1}^{n-1} \left(\lambda_i - \frac{2m}{n+p} \right) + (p+1) \frac{2m}{p+n} \\ &= 2m + (p-n-2) \frac{2m}{p+n}. \end{aligned}$$

Clearly $LE(G_1 \cup \bar{K}_p) = LE(G_2 \cup \bar{K}_p)$. \square

A result similar to Theorem 3.1 can be put forward for the graphs G_1 and G_2 for the Q -spectra in the following way.

Theorem 3.2. Let G_1 and G_2 be two connected non bipartite graphs having Q -spectra $0 < \mu_n^+ < \mu_{n-1}^+ \leq \dots \leq \mu_1^+$ and $0 < \lambda_n^+ < \lambda_{n-1}^+ \leq \dots \leq \lambda_1^+$, respectively. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_n^+, \lambda_n^+),$$

we have $LE^+(G_1 \cup \bar{K}_p) = LE^+(G_2 \cup \bar{K}_p)$.

Proof. The proof follows by the same argument as in Theorem 3.1. \square

For any two connected graphs with same number of vertices and edges, the next result gives the construction of connected graphs having the same L -energy.

Theorem 3.3. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$, with algebraic connectivities greater than one. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1} - 1, \lambda_{n-1} - 1),$$

we have $LE(\bar{G}_1 \vee K_p) = LE(\bar{G}_2 \vee K_p)$.

Proof. The L -spectra of K_p is 0, p ($(p-1)$ -times), therefore by Lemmas 2.3, 2.5, it follows that the L -spectra of graphs $\bar{G}_1 \vee K_p$ and $\bar{G}_2 \vee K_p$ is $p+n-\mu_1, p+n-\mu_2, \dots, p+n-\mu_{n-1}, (p+n)(p\text{-times}), 0$ and $p+n-\lambda_1, p+n-\lambda_2, \dots, p+n-\lambda_{n-1}, (p+n)(p\text{-times}), 0$, respectively, with average vertex degree $\frac{2m'}{n'} = p+n-1-\frac{2m}{p+n}$. So, for $i = 1, 2, \dots, n-1$, we have

$$p+n-\mu_i - \frac{2m'}{n'} = -\left(\mu_i - 1 - \frac{2m}{p+n}\right) \leq 0.$$

Similarly, $p+n-\lambda_i - \frac{2m'}{n'} \leq 0$. Therefore,

$$\begin{aligned} LE(\bar{G}_1 \vee K_p) &= \sum_{i=1}^{n-1} \left| p+n-\mu_i - \frac{2m'}{n'} \right| + p \left| p+n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right| \\ &= (n-p) \left(\frac{2m'}{n'} \right) + (p+n)(p-n+1). \end{aligned}$$

Also,

$$\begin{aligned} LE(\bar{G}_2 \vee K_p) &= \sum_{i=1}^{n-1} \left| p+n-\lambda_i - \frac{2m'}{n'} \right| + p \left| p+n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right| \\ &= (n-p) \left(\frac{2m'}{n'} \right) + (p+n)(p-n+1). \end{aligned}$$

Clearly, $LE(\bar{G}_1 \vee K_p) = LE(\bar{G}_2 \vee K_p)$. \square

Remark 3.4. Since the graph $\bar{G} \vee K_p$ is the complement of the graph $G \cup \bar{K}_p$, from Theorems 3.1 and 3.3, we have $LE(\bar{G}_1 \vee K_p) = LE(\bar{G}_2 \vee K_p)$ and $LE(G_1 \cup \bar{K}_p) = LE(G_2 \cup \bar{K}_p)$.

Thus Theorems 3.1 and 3.3 give families of pairs of graphs which are L -equienergetic as well as their complements are also L -equienergetic.

Corollary 3.5. Let $G_1(s, m)$ and $G_2(s, m)$ be two connected proper subgraphs of the complete graph K_n having L -spectra $0 = \mu_s < \mu_{s-1} \leq \dots \leq \mu_1$ and $0 = \lambda_s < \lambda_{s-1} \leq \dots \leq \lambda_1$, respectively, with algebraic connectivities greater than one. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{s-1} - 1, \lambda_{s-1} - 1),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (K_n - E(G_i)) \vee K_p$ for $i = 1, 2$.

Proof. The result can be proved by using the same argument as in [Theorem 3.3](#) and the fact that the L -spectra of $K_n - E(G_1)$ and $K_n - E(G_2)$ are $n - \mu_1, n - \mu_2, \dots, n - \mu_s, n((n-s-1)\text{-times}), 0$ and $n - \lambda_1, n - \lambda_2, \dots, n - \lambda_s, n((n-s-1)\text{-times}), 0$, respectively, with average vertex degree $\frac{2m'}{n'} = p + n - 1 - \frac{2m}{p+n}$. \square

If k is the first value of positive integer p satisfying the condition given in the above results, then every integer greater than k also satisfies this condition, hence we will obtain sequences of graphs having same L -energy (Q -energy). The next result gives another way of construction of graphs having the same L -energy.

Theorem 3.6. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq n$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(G_1 \vee \bar{K}_p) = LE(G_2 \vee \bar{K}_p)$.

Proof. The L -spectra of the graphs $G_1 \vee \bar{K}_p$ and $G_2 \vee \bar{K}_p$ (by [Lemma 2.3](#)) are $p + \mu_1, p + \mu_2, \dots, p + \mu_{n-1}, (p+n)((p-1)\text{-times}), 0$ and $p + \lambda_1, p + \lambda_2, \dots, p + \lambda_{n-1}, (p+n)((p-1)\text{-times}), 0$ respectively, with average vertex degree $\frac{2m'}{n'} = \frac{2m}{p+n} + \frac{2pn}{p+n}$. So, for $i = 1, 2, \dots, n-1$, we have

$$p + \mu_i - \frac{2m'}{n'} = \frac{p(p-n)}{p+n} + \mu_i - \frac{2m}{p+n} \geq 0.$$

Similarly, $p + \lambda_i - \frac{2m'}{n'} \geq 0$. Therefore,

$$\begin{aligned} LE(G_1 \vee \bar{K}_p) &= \sum_{i=1}^{n-1} \left| p + \mu_i - \frac{2m'}{n'} \right| + (p-1) \left| n - \frac{2m'}{n'} \right| + \left| p + n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right| \\ &= (p-n) \left(\frac{2m'}{n'} \right) + 2(m+n). \end{aligned}$$

Also,

$$\begin{aligned} LE(G_2 \vee \bar{K}_p) &= \sum_{i=1}^{n-1} \left| p + \lambda_i - \frac{2m'}{n'} \right| + (p-1) \left| n - \frac{2m'}{n'} \right| + \left| p + n - \frac{2m'}{n'} \right| + \left| 0 - \frac{2m'}{n'} \right| \\ &= (p-n) \left(\frac{2m'}{n'} \right) + 2(m+n). \end{aligned}$$

Clearly, $LE(G_1 \vee \bar{K}_p) = LE(G_2 \vee \bar{K}_p)$. \square

Remark 3.7. If in [Theorem 3.6](#), the complete graph \bar{K}_p is replaced by the graph $K_{p,p}$, we will obtain another family of L -equienergetic graphs and we have $LE(G_1 \vee \bar{K}_{p,p}) = LE(G_2 \vee \bar{K}_{p,p})$.

Theorem 3.8. Let $G_1(n, m)$ and $G_2(n, m)$ be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq n+4$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1 \cup K_p) = LE(\tilde{G}_2 \cup K_p)$.

Proof. The L -spectra of the graphs $\tilde{G}_1 \cup K_p$ and $\tilde{G}_2 \cup K_p$ (by [Lemmas 2.4, 2.5](#)) are $n - \mu_1, n - \mu_2, \dots, n - \mu_{n-1}, p((p-1)\text{-times}), 0, 0$ and $n - \lambda_1, n - \lambda_2, \dots, n - \lambda_{n-1}, p((p-1)\text{-times}), 0, 0$, respectively, with average vertex degree $\frac{2m'}{n'} = \frac{n^2+p^2}{p+n} - \frac{2m}{p+n} - 1$. So, for $i = 1, 2, \dots, n-1$, we have

$$n - \mu_i - \frac{2m'}{n'} = \frac{p(n-p)}{p+n} - \left(\mu_i - 1 - \frac{2m}{p+n} \right) \leq 0.$$

Similarly, $n - \lambda_i - \frac{2m'}{n'} = \frac{p(n-p)}{p+n} - (\lambda_i - 1 - \frac{2m}{p+n}) \leq 0$. Therefore,

$$\begin{aligned} LE(\bar{G}_1 \cup K_p) &= \sum_{i=1}^{n-1} \left| n + \mu_i - \frac{2m'}{n'} \right| + (p-1) \left| p - \frac{2m'}{n'} \right| + 2 \left| 0 - \frac{2m'}{n'} \right| \\ &= (n-1) \left(\frac{2m'}{n'} - n \right) + (p-1) \left| p - \frac{2m'}{n'} \right| + \frac{4m'}{n'} - 2m. \end{aligned}$$

Also,

$$\begin{aligned} LE(\bar{G}_2 \cup K_p) &= \sum_{i=1}^{n-1} \left| n + \lambda_i - \frac{2m'}{n'} \right| + (p-1) \left| p - \frac{2m'}{n'} \right| + 2 \left| 0 - \frac{2m'}{n'} \right| \\ &= (n-1) \left(\frac{2m'}{n'} - n \right) + (p-1) \left| p - \frac{2m'}{n'} \right| + \frac{4m'}{n'} - 2m \\ &= LE(\bar{G}_1 \cup K_p). \quad \square \end{aligned}$$

Remark 3.9. If $G_1(s, m)$ and $G_2(s, m)$ are two connected proper subgraphs of the complete graph K_n having L -spectra respectively $0 = \mu_s < \mu_{s-1} \leq \dots \leq \mu_1$ and $0 = \lambda_s < \lambda_{s-1} \leq \dots \leq \lambda_1$, with algebraic connectivities greater than one, then for a positive integer $p \geq n$ such that

$$\frac{2m}{p+n} < \min(\mu_{s-1} - 1, \lambda_{s-1} - 1),$$

we have $LE(\bar{G}_1) = LE(\bar{G}_2)$, where $G_i = (K_n - E(G_i)) \cup K_p$ for $i = 1, 2$.

Remark 3.10. Since the graph $\bar{G} \cup K_p$ is the complement of the graph $G \vee \bar{K}_p$, from Theorems 3.6 and 3.8 under the same conditions, it follows that $LE(\bar{G}_1 \cup K_p) = LE(\bar{G}_2 \cup K_p)$ and $LE(G_1 \vee \bar{K}_p) = LE(G_2 \vee \bar{K}_p)$.

Again, if k is the first value of positive integer p satisfying the condition given in Theorems 3.6, 3.8 and Remark 3.10, then every integer greater than k also satisfies this condition, hence we will obtain a sequence of L -equienergetic graph pairs. Note that such a sequence is possible as we are dealing with finite graphs.

4. Construction of L -equienergetic (Q -equienergetic) graphs by using product of graphs

In this section, we will use product of graphs together with the operations introduced in Section 3 to obtain the new families of L -equienergetic (Q -equienergetic) graphs from any given pair of connected graphs having same number of vertices and edges.

Theorem 4.1. Let $G_1(n, m)$ and $G_2(n, m)$ be two connected graphs having L -spectra, respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p > n$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (G_i \cup \bar{K}_p) \times K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs $G_1 \cup \bar{K}_p$ and $G_2 \cup \bar{K}_p$ are respectively $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = \mu_{n+1} = \dots = \mu_{p+n} = 0$ and $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n = \lambda_{n+1} = \dots = \lambda_{p+n} = 0$. Also the L -spectra of K_p are $p = \tau_1 = \tau_2 = \dots = \tau_{p-1}, \tau_p = 0$. So, by Lemma 2.1, the L -spectra of graphs \tilde{G}_1 and \tilde{G}_2 are respectively $\mu_i + \tau_j$ and $\lambda_i + \tau_j$, where $1 \leq i \leq p+n, 1 \leq j \leq p$, with average vertex degree $\frac{2m'}{n'} = \frac{2m}{n+p} + p - 1$. So, for $i = 1, 2, \dots, n-1$, we have

$$p + \mu_i - \frac{2m'}{n'} = \mu_i - \frac{2m}{n+p} + 1 > 0$$

and

$$\mu_i - \frac{2m'}{n'} = \mu_i - \frac{2m}{p+n} - p + 1 \leq 0.$$

Similarly, $p + \lambda_i - \frac{2m'}{n'} > 0$ and $\lambda_i - \frac{2m'}{n'} \leq 0$. Therefore,

$$\begin{aligned} LE(\tilde{G}_1) &= \sum_{i=1}^{p+n} \sum_{j=1}^p \left| \mu_i + \tau_j - \frac{2m'}{n'} \right| \\ &= (p-1) \sum_{i=1}^{p+n} \left| p + \mu_i - \frac{2m'}{n'} \right| + \sum_{i=1}^{p+n} \left| \mu_i - \frac{2m'}{n'} \right| \\ &= (p-1) \sum_{i=1}^{n-1} \left(p + \mu_i - \frac{2m'}{n'} \right) + (p-1) \sum_{i=n}^{p+n} \left| p - \frac{2m'}{n'} \right| + \sum_{i=1}^{p+n} \left(\frac{2m'}{n'} - \mu_i \right) \\ &= (p-1)(n-1) \left(p - \frac{2m'}{n'} \right) + (p-1)(p+n) \left| p - \frac{2m'}{n'} \right| + (p+n) \frac{2m'}{n'} + 2m(p-2). \end{aligned}$$

Also

$$\begin{aligned} LE(\tilde{G}_2) &= \sum_{i=1}^{p+n} \sum_{j=1}^p \left| \lambda_i + \tau_j - \frac{2m'}{n'} \right| \\ &= (p-1)(n-1) \left(p - \frac{2m'}{n'} \right) + (p-1)(p+n) \left| p - \frac{2m'}{n'} \right| + (p+n) \frac{2m'}{n'} + 2m(p-2). \end{aligned}$$

Hence the result follows. \square

A result similar to [Theorem 4.1](#) can be put forward for the connected graphs G_1 and G_2 for the Q -spectra in the following way.

Theorem 4.2. Let $G_1(n, m)$ and $G_2(n, m)$ be two connected graphs having Q -spectra, respectively $0 < \mu_n^+ \leq \mu_{n-1}^+ \leq \dots \leq \mu_1^+$ and $0 < \lambda_n^+ \leq \lambda_{n-1}^+ \leq \dots \leq \lambda_1^+$, with $\mu_n^+, \lambda_n^+ > 1$. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_n^+ - 1, \lambda_n^+ - 1),$$

we have $LE^+((G_1 \cup \bar{K}_p) \times K_p) = LE^+((G_2 \cup \bar{K}_p) \times K_p)$.

Proof. The Q -spectra of the graphs $(G_1 \cup \bar{K}_p) \times K_p$ and $(G_2 \cup \bar{K}_p) \times K_p$ are (by [Lemmas 2.1](#) and [2.4](#)) $2p-2 + \mu_1^+, 2p-2 + \mu_2^+, \dots, 2p-2 + \mu_n^+, (2p-2)(p\text{-times}), \{p-2 + \mu_1^+, \dots, p-2 + \mu_n^+\}(\text{each}(p-1)\text{-times}), (p-2)(p(p-1)\text{-times})$ and $2p-2 + \lambda_1^+, 2p-2 + \lambda_2^+, \dots, 2p-2 + \lambda_n^+, (2p-2)(p\text{-times}), \{p-2 + \lambda_1^+, \dots, p-2 + \lambda_n^+\}(\text{each}(p-1)\text{-times}), (p-2)(p(p-1)\text{-times})$, respectively, with average vertex degree $\frac{2m'}{n'} = \frac{2m}{n+p} + p-1$. So for $i = 1, 2, \dots, n-1$, we have

$$2p-2 + \mu_i^+ - \frac{2m'}{n'} = p + \mu_i^+ - \frac{2m}{n+p} - 1 > 0$$

and

$$p-2 + \mu_i^+ - \frac{2m'}{n'} = \mu_i^+ - \frac{2m}{p+n} - 1 > 0.$$

Similarly, $2p-2 + \lambda_i^+ - \frac{2m'}{n'} > 0$ and $p-2 + \mu_i^+ - \frac{2m'}{n'} > 0$.

The results now follow by proceeding in the same way as in [Theorem 4.1](#). \square

[Theorem 4.2](#) is true for connected graphs G_1 and G_2 having algebraic connectivity greater than one. However, if we consider graph $(G \cup \bar{K}_p) \times K_{p,p}$ in place of the graph $(G \cup \bar{K}_p) \times K_p$, it holds for any connected pair of graphs as seen in the following.

Corollary 4.3. Let G_1 and G_2 be two connected graphs having Q -spectra respectively $0 < \mu_n^+ \leq \mu_{n-1}^+ \leq \dots \leq \mu_1^+$ and $0 < \lambda_n^+ \leq \lambda_{n-1}^+ \leq \dots \leq \lambda_1^+$. For a positive integer $p \geq 2n$ such that

$$\frac{2m}{p+n} < \min(\mu_n^+, \lambda_n^+),$$

we have $LE^+((G_1 \cup \bar{K}_p) \times K_{p,p}) = LE^+((G_2 \cup \bar{K}_p) \times K_p)$.

Proof. The proof follows by the same argument as in Theorem 4.2 and the fact that the Q -spectra of the graph $K_{p,p}$ is $p, \frac{p}{2}((p-2)\text{-times}), 0$. \square

The following is a direct consequence of Theorem 4.1.

Corollary 4.4. Let $G_1(s, m)$ and $G_2(s, m)$ be two connected proper subgraphs of the complete graph K_n having L -spectra respectively $0 = \mu_s < \mu_{s-1} \leq \dots \leq \mu_1$ and $0 = \lambda_s < \lambda_{s-1} \leq \dots \leq \lambda_1$, with algebraic connectivities greater than two. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1} - 2, \lambda_{n-1} - 2),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $G_i = ((K_n - E(G_i)) \cup K_p) \times K_p$ for $i = 1, 2$.

Theorem 4.5. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq 2n$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (\bar{G}_i \cup K_p) \times K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs $\bar{G}_1 \cup K_p$ and $\bar{G}_2 \cup K_p$ are respectively as $\gamma_1, \gamma_2, \dots, \gamma_{p+n}$ and $\theta_1, \theta_2, \dots, \theta_{p+n}$, where

$$\gamma_i = \begin{cases} n - \mu_1, & \text{if } 1 \leq i \leq n-1; \\ 0, & \text{if } i = n, p+n; \\ p, & \text{if } n+1 \leq i \leq p+n-1 \end{cases}$$

and

$$\theta_i = \begin{cases} n - \lambda_i, & \text{if } 1 \leq i \leq n-1; \\ 0, & \text{if } i = n, p+n; \\ p, & \text{if } n+1 \leq i \leq p+n-1. \end{cases}$$

Also L -spectra of K_p is $p = \tau_1 = \tau_2 = \dots = \tau_{p-1}, 0 = \tau_p$. So by Lemma 2.1, the L -spectra of \tilde{G}_1 and \tilde{G}_2 are $\gamma_i + \tau_j$ and $\theta_i + \tau_j$, where $1 \leq i \leq p+n, 1 \leq j \leq p$ with average vertex degree $\frac{2m'}{n'} = p - 2 - \frac{2m}{n+p} + \frac{p^2+n^2}{p+n}$. For $i = 1, 2, \dots, n-1$, we have

$$n - \mu - i - \frac{2m'}{n'} = \frac{2(p+n) - 2p^2}{p+n} + \frac{2m}{p+n} - \mu_i \leq 0$$

and

$$p+n - \mu_i - \frac{2m'}{n'} = \frac{p(n-p) + 2(p+n)}{p+n} + \frac{2m}{p+n} - \mu_i \leq 0.$$

Similarly, $n - \lambda_i - \frac{2m'}{n'} \leq 0$ and $p+n - \lambda_i - \frac{2m'}{n'} \leq 0$.

It is now easy to see that $LE(\tilde{G}_1) = LE(\tilde{G}_2)$. \square

Now we will use Cartesian product to construct families of connected graphs having same L -energy.

Theorem 4.6. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$, with algebraic connectivities greater than two. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1} - 2, \lambda_{n-1} - 2),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (\bar{G}_i \vee K_p) \times K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs $\bar{G}_1 \vee K_p$ and $\bar{G}_2 \vee K_p$ are respectively as $\gamma_1, \gamma_2, \dots, \gamma_{p+n}$ and $\theta_1, \theta_2, \dots, \theta_{p+n}$, where

$$\gamma_i = \begin{cases} p+n-\mu_1, & \text{if } 1 \leq i \leq n-1; \\ p+n, & \text{if } n \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n \end{cases}$$

and

$$\theta_i = \begin{cases} p+n-\lambda_i, & \text{if } 1 \leq i \leq n-1; \\ p+n, & \text{if } n \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n. \end{cases}$$

Also L -spectra of K_p is $p = \tau_1 = \tau_2 = \dots = \tau_{p-1}, 0 = \tau_p$. So by Lemma 2.1, the L -spectra of \tilde{G}_1 and \tilde{G}_2 are $\gamma_i + \tau_j$ and $\theta_i + \tau_j$, where $1 \leq i \leq p+n, 1 \leq j \leq p$ with average vertex degree $\frac{2m'}{n'} = 2p+n-2 - \frac{2m}{n+p}$. For $i = 1, 2, \dots, n-1$, we have

$$2p+n-\mu_i - \frac{2m'}{n'} = -\left(\mu_i - \frac{2m}{p+n} - 2\right) \leq 0$$

and

$$p+n-\mu_i - \frac{2m'}{n'} = \frac{2m}{p+n} - \mu_i + 2 - p < 0.$$

Similarly, $2p+n-\lambda_i - \frac{2m'}{n'} \leq 0$ and $p+n-\lambda_i - \frac{2m'}{n'} < 0$.

It is now easy to see that $LE(\tilde{G}_1) = LE(\tilde{G}_2)$. \square

The following is a direct consequence of Theorem 4.6.

Corollary 4.7. Let $G_1(s, m)$ and $G_2(s, m)$ be two connected proper subgraphs of the complete graph K_n having L -spectra respectively $0 = \mu_s < \mu_{s-1} \leq \dots \leq \mu_1$ and $0 = \lambda_s < \lambda_{s-1} \leq \dots \leq \lambda_1$, with algebraic connectivities greater than two. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1} - 2, \lambda_{n-1} - 2),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $(G_i = (K_n - E(G_1)) \vee K_p) \times K_p$ for $i = 1, 2$.

Theorem 4.8. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq n+4$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (G_i \vee \bar{K}_p) \times K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs $G_1 \vee \bar{K}_p$ and $G_2 \vee \bar{K}_p$ are respectively as $\gamma_1, \gamma_2, \dots, \gamma_{p+n}$ and $\theta_1, \theta_2, \dots, \theta_{p+n}$, where

$$\gamma_i = \begin{cases} p+\mu_1, & \text{if } 1 \leq i \leq n-1; \\ p+n, & \text{if } i = n; \\ n, & \text{if } n+1 \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n \end{cases}$$

and

$$\theta_i = \begin{cases} p + \lambda_i, & \text{if } 1 \leq i \leq n-1; \\ p + n, & \text{if } i = n; \\ n, & \text{if } n+1 \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n. \end{cases}$$

Also, L -spectra of K_p is $p = \tau_1 = \tau_2 = \dots = \tau_{p-1}, 0 = \tau_p$. So, by [Lemma 2.1](#), the L -spectra of \tilde{G}_1 and \tilde{G}_2 is $\gamma_i + \tau_j$ and $\theta_i + \tau_j$, where $1 \leq i \leq p+n, 1 \leq j \leq p$, with average vertex degree $\frac{2m'}{n'} = p-1 + \frac{2m}{n+p} + \frac{2pn}{p+n}$. For $i = 1, 2, \dots, n-1$, we have

$$2p + \mu_i - \frac{2m'}{n'} = \frac{p(p-n)}{p+n} + \mu_i - \frac{2m}{p+n} + 1 \geq 0$$

and

$$p + \mu_i - \frac{2m'}{n'} = \frac{(p+n)\mu_i - 2pn}{p+n} + \frac{(p+n) - 2m}{p+n} \leq 0.$$

Similarly, $2p + \lambda_i - \frac{2m'}{n'} \geq 0$ and $p + \lambda_i - \frac{2m'}{n'} \leq 0$.

Therefore,

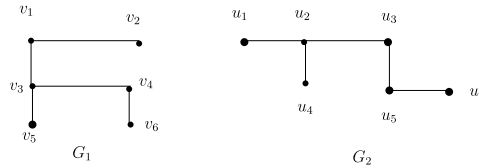
$$\begin{aligned} LE(\tilde{G}_1) &= \sum_{i=1}^{p+n} \sum_{j=1}^p \left| \gamma_i + \tau_j - \frac{2m'}{n'} \right| \\ &= (p-1) \sum_{i=1}^{n-1} \left(2p + \mu_i - \frac{2m'}{n'} \right) + (p-1) \sum_{i=n+1}^{p+n-1} \left| p + n - \frac{2m'}{n'} \right| \\ &\quad + (p-1) \left| 2p + n - \frac{2m'}{n'} \right| + \sum_{i=1}^{n+p} \left| \gamma_i - \frac{2m'}{n'} \right| + (p-1) \left| p - \frac{2m'}{n'} \right| \\ &= (p-1)(n-1) \left(2p - \frac{2m'}{n'} \right) + 2m(p-2) + (p-1)^2 \left| p + n - \frac{2m'}{n'} \right| \\ &\quad + (p-1) \left| 2p + n - \frac{2m'}{n'} \right| + (p-1) \left| p - \frac{2m'}{n'} \right| + (n-1) \left(\frac{2m'}{n'} - p \right) \\ &\quad + (p-1) \left| n - \frac{2m'}{n'} \right| + \left| p + n - \frac{2m'}{n'} \right| + \frac{2m'}{n'}. \end{aligned}$$

Also,

$$\begin{aligned} LE(\tilde{G}_2) &= \sum_{i=1}^{p+n} \sum_{j=1}^p \left| \theta_i + \tau_j - \frac{2m'}{n'} \right| \\ &= (p-1)(n-1) \left(2p - \frac{2m'}{n'} \right) + 2m(p-2) + (p-1)^2 \left| p + n - \frac{2m'}{n'} \right| \\ &\quad + (p-1) \left| 2p + n - \frac{2m'}{n'} \right| + (p-1) \left| p - \frac{2m'}{n'} \right| + (n-1) \left(\frac{2m'}{n'} - p \right) \\ &\quad + (p-1) \left| n - \frac{2m'}{n'} \right| + \left| p + n - \frac{2m'}{n'} \right| + \frac{2m'}{n'}. \end{aligned}$$

Clearly, $LE(\tilde{G}_1) = LE(\tilde{G}_2)$. \square

Now, we have the following observation.

Fig. 1. Example of graphs for which $LE(G_1 \times K_p) \neq LE(G_2 \times K_p)$.

Corollary 4.9. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq n$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE((G_1 \vee K_p) \cup \bar{K}_p) = LE((G_2 \vee K_p) \cup \bar{K}_p)$.

Proof. The L -spectra of the graphs $(G_1 \vee K_p) \cup \bar{K}_p$ and $(G_2 \vee K_p) \cup \bar{K}_p$ are respectively $p + \mu_1, p + \mu_2, \dots, p + \mu_{n-1}, (p+n)(p\text{-times}), 0(p\text{-times})$ and $p + \lambda_1, p + \lambda_2, \dots, p + \lambda_{n-1}, (p+n)(p\text{-times}), 0(p\text{-times})$, with average vertex degree $\frac{2m'}{n'} = \frac{2pn}{2p+n} + \frac{2m}{2p+n} + \frac{p(p-1)}{2p+n}$. So for $i = 1, 2, \dots, n-1$, we have

$$p+n - \frac{2m'}{n'} = \frac{p(p-n+1)}{2p+n} + \mu_i - \frac{2m}{n+2p} \geq 0$$

and

$$p + \lambda_i - \frac{2m'}{n'} \geq 0.$$

Therefore it can be seen that $LE((G_1 \vee K_p) \cup \bar{K}_p) = LE((G_2 \vee K_p) \cup \bar{K}_p)$. \square

For any two graphs G_1 and G_2 having the same number of vertices and edges, it is not always true that $LE(G_1 \times K_p) = LE(G_2 \times K_p)$. As an example consider the graphs G_1 and G_2 (Fig. 1). By direct calculation it can be seen that for $p = 4$, $LE(G_1 \times K_p) = 41.70818 \neq 42.05078 = LE(G_2 \times K_p)$; for $p = 7$, $LE(G_1 \times K_p) = 83.4164 \neq 84.1016 = LE(G_2 \times K_p)$; and for $p = 6$, $LE(G_1 \times K_p) = 69.5139 \neq 70.0849 = LE(G_2 \times K_p)$. However, for a certain class of graphs, whose algebraic connectivity when added to one becomes greater than or equal to average vertex degree, the above equality holds for $p > n$, which can be seen in the following result.

Theorem 4.10. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$, with $\min(\mu_{n-1}, \lambda_{n-1}) \geq \frac{2m}{n} + 1$, then for $p > n$, $LE(G_1 \times K_p) = LE(G_2 \times K_p)$.

Proof. The L -spectra of the graphs $G_1 \times K_p$ and $G_2 \times K_p$ are respectively $\mu_i + \tau_j$ and $\lambda_i + \tau_j$, where $1 \leq i \leq n$ and $1 \leq j \leq p$ and τ_j is as in Theorem 4.8. With average degree $\frac{2m'}{n'} = \frac{2m}{n} + p - 1$. Therefore,

$$\begin{aligned} LE(G_1 \times K_p) &= \sum_{i=1}^n \sum_{j=1}^p \left| \mu_i + \tau_j - \frac{2m'}{n'} \right| \\ &= (p-1) \sum_{i=1}^n \left| \mu_i + p - \frac{2m}{n} - p + 1 \right| + \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} - p + 1 \right| \\ &= (p-1) \sum_{i=1}^n \left(\mu_i - \frac{2m}{n} + 1 \right) + \sum_{i=1}^n \left(\frac{2m}{n} + p - 1 - \mu_i \right) \\ &= 2m(p-2) + n(p-2) \left(1 - \frac{2m}{n} \right) + pn. \end{aligned}$$

Also

$$\begin{aligned} LE(G_2 \times K_p) &= \sum_{i=1}^n \sum_{j=1}^p \left| \lambda_i + \tau_j - \frac{2m'}{n'} \right| \\ &= 2m(p-2) + n(p-2) \left(1 - \frac{2m}{n} \right) + pn \\ &= LE(G_1 \times K_p). \quad \square \end{aligned}$$

The following result constructs a sequence of graph pairs having same L -energy.

Theorem 4.11. Let $G_1(n, m)$ and $G'_1(n, m)$ be two L -equienergetic graphs having L -spectra respectively $0 = \mu_n \leq \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \mu'_n \leq \mu'_{n-1} \leq \dots \leq \mu'_1$, and let $G_2(n, m)$ and $G'_2(n, m)$ be another pair of L -equienergetic graphs having L -spectra respectively $0 = \lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1$ and $0 = \lambda'_n \leq \lambda'_{n-1} \leq \dots \leq \lambda'_1$, then $LE(G_1 \vee G_2) = LE(G'_1 \vee G'_2)$.

Proof. If $H = G_1 \vee G_2$ and $K = G'_1 \vee G'_2$, then the L -spectra of H and K are respectively $2n, n + \mu_1, \dots, n + \mu_{n-1}, n + \lambda_1, n + \lambda_2, \dots, n + \lambda_{n-1}, 0$ and $2n, n + \mu'_1, \dots, n + \mu'_{n-1}, n + \lambda'_1, n + \lambda'_2, \dots, n + \lambda'_{n-1}, 0$, with average vertex degree $\frac{2m'}{n'} = \frac{2m+n^2}{n}$. Since (G_1, G'_1) and (G_2, G'_2) are L -equienergetic graph pairs, therefore $LE(G_1) = LE(G'_1)$ and $LE(G_2) = LE(G'_2)$. Now,

$$\begin{aligned} LE(H) &= \left| 2n - \frac{2m+n^2}{n} \right| + \sum_{i=1}^{n-1} \left| n + \mu_i - \frac{2m+n^2}{n} \right| \\ &\quad + \sum_{i=1}^{n-1} \left| n + \lambda_i - \frac{2m+n^2}{n} \right| + \left| 0 - \frac{2m+n^2}{n} \right| \\ &= 2n + LE(G_1) + LE(G_2) - \frac{4m}{n} \end{aligned}$$

and

$$\begin{aligned} LE(K) &= \left| 2n - \frac{2m+n^2}{n} \right| + \sum_{i=1}^{n-1} \left| n + \mu'_i - \frac{2m+n^2}{n} \right| + \sum_{i=1}^{n-1} \left| n + \lambda'_i - \frac{2m+n^2}{n} \right| + \left| 0 - \frac{2m+n^2}{n} \right| \\ &= 2n + LE(G'_1) + LE(G'_2) - \frac{4m}{n}. \end{aligned}$$

Clearly the result follows. \square

The next result is a generalization of [Theorem 4.11](#).

Theorem 4.12. Let $(G_1, G'_1), (G_2, G'_2), \dots, (G_k, G'_k)$ be k L -equienergetic graph pairs with same number of vertices n and edges m , then $LE(G_1 \vee G_2 \vee \dots \vee G_k) = LE(G'_1 \vee G'_2 \vee \dots \vee G'_k)$.

Proof. Let $0 = \mu_{ni} \leq \mu_{(n-1)i} \leq \dots \leq \mu_{1i}$ and $0 = \mu'_{ni} \leq \mu'_{(n-1)i} \leq \dots \leq \mu'_{1i}$ be the L -spectra of the graphs G_i and G'_i , respectively. Then applying [Lemma 2.3](#) repeatedly, we find that the L -spectra of the graphs $(G_1 \vee G_2 \vee \dots \vee G_k)$ and $(G'_1 \vee G'_2 \vee \dots \vee G'_k)$ are $kn((k-1)\text{-times}), n(k-1) + \mu_{1i}, \dots, n(k-1) + \mu_{(n-1)i}, 0$ and $kn((k-1)\text{-times}), n(k-1) + \mu'_{1i}, \dots, n(k-1) + \mu'_{(n-1)i}, 0$, respectively, with average vertex degree $\frac{2m'}{n'} = \frac{2m+n^2(k-1)}{n}$.

Therefore,

$$\begin{aligned} LE((G_1 \vee G_2 \vee \dots \vee G_k)) &= (k-1) \left| kn - \frac{(k-1)2m+n^2}{n} \right| \\ &\quad + \sum_{i=1}^k \sum_{j=1}^n \left| n(k-1) + \mu_{ji} - \frac{2m+n^2(k-1)}{n} \right| + \left| 0 - \frac{2m+n^2(k-1)}{n} \right| \end{aligned}$$

$$\begin{aligned}
&= (k-1) \left(kn - \frac{2m + n^2(k-1)}{n} \right) + \sum_{i=1}^k \left(LE(G_i) - \frac{2m}{n} \right) + \frac{2m + n^2(k-1)}{n} \\
&= \sum_{i=1}^k LE(G_i) + 2n(k-1) - (2k-2) \frac{2m}{n}.
\end{aligned}$$

Also,

$$\begin{aligned}
LE((G'_1 \vee G'_2 \vee \dots \vee G'_k)) &= (k-1) \left| kn - \frac{2m + n^2(k-1)}{n} \right| \\
&\quad + \sum_{i=1}^k \sum_{j=1}^n \left| n(k-1) + \mu'_{ji} - \frac{2m + n^2(k-1)}{n} \right| + \left| 0 - \frac{2m + n^2(k-1)}{n} \right| \\
&= \sum_{i=1}^k LE(G'_i) + 2n(k-1) - (2k-2) \frac{2m}{n}.
\end{aligned}$$

Clearly, the result follows. \square

Using the Kronecker product of graphs, we now give another construction of sequences of L -equienergetic (Q -equienergetic) graphs from a given pair of connected graphs.

Theorem 4.13. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer $p \geq n$ such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (G_i \vee \bar{K}_p) \otimes K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs $G_1 \vee \bar{K}_p$ and $G_2 \vee \bar{K}_p$ are respectively $\gamma_1, \gamma_2, \dots, \gamma_{p+n}$ and $\theta_1, \theta_2, \dots, \theta_{p+n}$, where

$$\gamma_i = \begin{cases} p + \mu_1, & \text{if } 1 \leq i \leq n-1; \\ p + n, & \text{if } i = n; \\ n, & \text{if } n+1 \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n \end{cases}$$

and

$$\theta_i = \begin{cases} p + \lambda_i, & \text{if } 1 \leq i \leq n-1; \\ p + n, & \text{if } i = n; \\ n, & \text{if } n+1 \leq i \leq p+n-1; \\ 0, & \text{if } i = p+n. \end{cases}$$

Also, L -spectra of K_p is $p = \tau_1 = \tau_2 = \dots = \tau_{p-1}, 0 = \tau_p$. So the L -spectra of \tilde{G}_1 and \tilde{G}_2 are $\gamma_i \tau_j$ and $\theta_i \tau_j$, where $1 \leq i \leq p+n, 1 \leq j \leq p$, with average vertex degree $\frac{2m'}{n'} = \frac{2m(p-1)}{n+p} + \frac{2pn(p-1)}{p+n}$. For $i = 1, 2, \dots, n-1$, we have

$$p(p + \mu_i) - \frac{2m'}{n'} = p\mu_i - \frac{2m(p-1)}{p+n} + \frac{p(p^2 - pn + 2n)}{p+n} \geq 0.$$

Similarly $p(p + \lambda_i) - \frac{2m'}{n'} \geq 0$. It is now easy to see that $LE(\tilde{G}_1) = LE(\tilde{G}_2)$. \square

Theorems 4.14 and **4.15** are concerned with the construction of disconnected graphs having same L -energy (Q -energy) from a given pair of connected graphs.

Theorem 4.14. Let G_1 and G_2 be two connected graphs having L -spectra respectively $0 = \mu_n < \mu_{n-1} \leq \dots \leq \mu_1$ and $0 = \lambda_n < \lambda_{n-1} \leq \dots \leq \lambda_1$. For a positive integer p such that

$$\frac{2m}{p+n} < \min(\mu_{n-1}, \lambda_{n-1}),$$

we have $LE(\tilde{G}_1) = LE(\tilde{G}_2)$, where $\tilde{G}_i = (G_i \cup \bar{K}_p) \otimes K_p$, $i = 1, 2$.

Proof. The L -spectra of the graphs \tilde{G}_1 and \tilde{G}_2 are respectively given as $p\mu_1, p\mu_2, \dots, p\mu_{n-1}$ (each $(p-1)$ -times), $0((p^2 + p + n - 1)$ -times) and $p\lambda_1, p\lambda_2, \dots, p\lambda_{n-1}$ (each $(p-1)$ -times), $0((p^2 + p + n - 1)$ -times), with average vertex degree $\frac{2m'}{n'} = \frac{2m(p-1)}{p+n}$. So, for $i = 1, 2, \dots, n-1$, we have

$$p\mu_i - \frac{2m'}{n'} \geq 0$$

and

$$p\lambda_i - \frac{2m'}{n'} \geq 0.$$

Now, it is easy to see that $LE(\tilde{G}_1) = LE(\tilde{G}_2)$. \square

Theorem 4.15. Let G_1 and G_2 be two connected graphs having Q -spectra respectively $0 < \mu_n^+ < \mu_{n-1}^+ \leq \dots \leq \mu_1^+$ and $0 < \lambda_n^+ < \lambda_{n-1}^+ \leq \dots \leq \lambda_1^+$. For a positive integer $p > n$ such that

$$\frac{2m}{p+n} < \min(\mu_n^+, \lambda_n^+),$$

we have $LE^+((G_1 \cup \bar{K}_p) \otimes K_p) = LE^+((G_2 \cup \bar{K}_p) \otimes K_p)$.

Proof. The proof follows from Theorem 4.14 and the fact that the Q -spectra of K_p is $2p-2$, $p-2((p-1)$ -times). \square

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